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Optimal Shape Design Problems for a Class of Systems Described by Parabolic Hemivariational Inequality

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Abstract. Optimal shape design problems for systems governed by a parabolic hemivariational inequality are considered. A general existence result for this problem is established by the mapping method.

Key words: Optimal shape design, Hemivariational inequality, Lower semicontinuity, Clarke subdifferential.

1. Introduction

In this paper, we study the existence problem of optimal shape design for a parabolic hemivariational inequality $(PHVI)$ with a general cost functional of integral form. In our formulation it is a control problem, where $(PHVI)$ corresponds to a "state equation" and the controls are sets from a family $\mathcal{O}^{k,\infty}$ of admissible shapes (see Section 2).

So the optimal shape design problem $(OSDP)$ is of the form:

$$\begin{cases} \text{Find } \Omega^* \in \mathcal{B} \text{ and } u^* \in S(\Omega^*) \text{ such that} \\ J(\Omega^*, u^*) = \min\limits_{\Omega \in \mathcal{B}} \min\limits_{u \in S(\Omega)} J(\Omega, u), \end{cases} \tag{1}$$

where controls belong to $\mathcal{B}$, which is a bounded, closed subset of a family $\mathcal{O}^{k,\infty}$. Here we use the mapping method, introduced by Murat and Simon in [15] and [16], which provides us with an appropriate topology on $\mathcal{B}$. The functions u are taken from the set $S(\Omega)$ of the solutions to $(PHVI)$ which, in turn, is formulated as follows

$$\begin{cases} \text{find } u \in \mathcal{W} \text{ such that} \\ (u'(t), v - u(t))_{V' \times V} + a(u(t), v - u(t)) + \int_\Omega j^0(u(t), v - u(t))\, dx \geq \\ \qquad \geq (f(t), v - u(t))_{V' \times V}, \quad \forall v \in V, \text{ a.e. } t \in (0, I), \\ u(0) = \psi. \end{cases} \tag{2}$$

Above $a(\cdot,\cdot)$ is a bilinear form on $V = H^1(\Omega)$, and j^0 denotes the Clarke's directional derivative of a locally Lipschitz function $j : \mathbb{R} \to \mathbb{R}$ whose subdifferential

∂j describes a nonmonotone, nonconvex and possibly multivalued law in Ω. The existence result for $(PHVI)$ was obtained by Miettinen [13].

In this paper we present an existence result for $(OSDP)$ for systems governed by $(PHVI)$ (see Theorem 3). On one hand, our theorem generalizes to the case of parabolic hemivariational inequalities, the result on similar optimal shape design problems for variational inequality obtained by Liu and Rubio [12, Part 2]. On the other hand, it extends to parabolic case the existence result for $(OSDP)$ with hemivariational inequality for elliptic case proved by Denkowski and Migórski [7].

For the applications of our result we refer to [18], where some temperature control problems in heat conduction are considered. These problems were originally studied by Duvaut and Lions [5], where semipermeability relations of monotone type led to systems governed by variational inequalities. In [18] some generalizations of these problems, with not necessarily monotone semipermeability relations, are given. More precisely, the problem of regulating the temperature to deviate as little as possible from the given interval is considered. In this case the system is governed by a hemivariational inequality of parabolic type.

For optimal control problem for elliptic hemivariational inequalities, we refer to related papers [9] and [14]. However, both these papers deal with the situation in which the controls appear in the right hand side of the inequality and in the bilinear form. The existence of optimal solutions is obtained and the relation between the original problem and the finite dimensional one is investigated.

The organization of this paper is as follows. In Section 2 we recall the notions and basic facts on the mapping method, while in Section 3 we introduce some functional spaces needed in the sequel. In Section 4 we study $(PHVI)$ of the form (1). For such problem we prove the closedness of the graph of the multifunction $\mathcal{B} \ni \Omega \to S(\Omega)$ (in suitable topologies), as well as, we show some a priori estimates for the solutions of $(PHVI)$. This facts are crucial to get our main result on the existence of solutions to $(OSDP)$ which is formulated and proved in Section 5. Section 6 gives some final comments on the obtained result, shows some of its applications and indicates the possible ways of its generalization.

2. The mapping method

In this section we recall notation and basic results on the mapping method which were established by Murat and Simon in [15]. We keep the notation of [7].

Let C be a bounded open subset of $\mathbb{R}^N$ with a boundary ∂C of class $W^{i,\infty}$, $i \geq 1$ and such that $int\,\overline{C} = C$. Then, following [15], [12], [7], we introduce, for $k \geq 1$, the following spaces

$$W^{k,\infty}(\mathbb{R}^N, \mathbb{R}^N) = \left\{ \varphi \,|\, D^\alpha \varphi \in L^\infty(\mathbb{R}^N, \mathbb{R}^N) \text{ for all } \alpha, 0 \leq |\alpha| \leq k \right\},$$

where derivatives $D^\alpha\varphi$ are understood in the distributional sense. By $\mathcal{O}^{k,\infty}$ we will denote the space of bounded open sets of $\mathbb{R}^N$ which are isomorphic with C, i.e.

$$\mathcal{O}^{k,\infty} = \{\Omega \mid \Omega = T(C),\ T \in \mathcal{F}^{k,\infty}\},$$

where $\mathcal{F}^{k,\infty}$ is the space of regular bijections in $\mathbb{R}^N$ defined by

$$\mathcal{F}^{k,\infty} = \{T\colon \mathbb{R}^N \to \mathbb{R}^N \mid T \text{ is bijective and } T, T^{-1} \in \mathcal{V}^{k,\infty}\},$$

$$\mathcal{V}^{k,\infty} = \{T\colon \mathbb{R}^N \to \mathbb{R}^N \mid T - I \in W^{k,\infty}(\mathbb{R}^N, \mathbb{R}^N)\}.$$

In other words $\mathcal{F}^{k,\infty}$ represents the set of essentially bounded perturbations (with essentially bounded derivatives) of identity in $\mathbb{R}^N$. It can be seen (see [15]) that if C has a $W^{k,\infty}$ boundary, then every set $\Omega \in \mathcal{O}^{k,\infty}$ also has the boundary of class $W^{k,\infty}$. Endowing the space $W^{k,\infty}(\mathbb{R}^N, \mathbb{R}^N)$ with the norm

$$\|\varphi\|_{k,\infty} = \operatorname*{ess\,sup}_{x\in\mathbb{R}^N} \left(\sum_{0\le|\alpha|\le k} |D^\alpha\varphi|^2_{\mathbb{R}^N} \right)^{\frac{1}{2}},$$

we define on $\mathcal{O}^{k,\infty} \times \mathcal{O}^{k,\infty}$ a function

$$\delta_{k,\infty}(\Omega_1, \Omega_2) = \inf_{T\in\mathcal{F}^{k,\infty}, T(\Omega_1)=\Omega_2} \left(\|T - I\|_{k,\infty} + \|T^{-1} - I\|_{k,\infty} \right).$$

The mapping $\delta_{k,\infty}$ is a pseudo-distance on $\mathcal{O}^{k,\infty}$ since it does not satisfy the triangle inequality (see Section 2.4 of [15]). From Proposition 2.3, Theorem 2.2 and Theorem 2.4 of [15], we have

THEOREM 1. *Let $k \ge 1$. Then*
(a) There exists a positive constant μ_k such that $d_{k,\infty}$ defined by $d_{k,\infty} = \sqrt{\delta_{k,\infty} \wedge \mu_k}$ is a metric on $\mathcal{O}^{k,\infty}$.
(b) The space $\left(\mathcal{O}^{k,\infty}, d_{k,\infty}\right)$ is a complete metric space.
(c) If $k \ge 2$, then the injection from $\mathcal{O}^{k,\infty}$ into $\mathcal{O}^{k-1,\infty}$ is compact. More precisely, if $k \ge 2$ and $\mathcal{B}$ is a bounded (in $\delta_{k,\infty}$), closed subset of $\mathcal{O}^{k,\infty}$, then for any sequence $\{\Omega_m\} \subset \mathcal{B}$, there exist a subsequence $\{\Omega_{m\nu}\}$ of $\{\Omega_m\}$ and a set $\Omega \in \mathcal{B}$ such that $\Omega_{m\nu} \to \Omega$ in $\mathcal{O}^{k-1,\infty}$.

REMARK 1. It is known (cf. Section 2 in [15]) that $\Omega_m \to \Omega$ in $\mathcal{O}^{k,\infty}$ iff there exist T_m and T in $\mathcal{F}^{k,\infty}$ such that $T_m(C) = \Omega_m$, $T(C) = \Omega$ and $T_m \to T$, $T_m^{-1} \to T^{-1}$ in $W^{k,\infty}(\mathbb{R}^N, \mathbb{R}^N)$.

Some other facts on the mapping method, are summarized in the following lemma.

LEMMA 1. *Let $k \geq 1$. Then*
(a) If $T \in \mathcal{F}^{1,\infty}$, $\Omega = T(C)$, then $u \in L^2(\Omega)$ iff $u \circ T \in L^2(C)$; $u \in H^1(\Omega)$ iff $u \circ T \in H^1(C)$. Moreover, if $u_m \to u$ in $H^1(\Omega)$ (or in $H^1(C)$) and $T \in \mathcal{F}^{k,\infty}$, then $u_m \circ T \to u \circ T$ in $H^1(C)$ (or $u_m \circ T^{-1} \to u \circ T^{-1}$ in $H^1(\Omega)$).
(b) Let $u \in H^l(\mathbb{R}^N)$ with $l = 0$ or 1. Then the mapping $T \mapsto u \circ T$ is continuous from $\mathcal{V}^{k,\infty}$ to $H^l(\mathbb{R}^N)$ at every point $T \in \mathcal{F}^{k,\infty}$.
(c) The following mappings are continuous

$$T \mapsto J_T^{-1} \text{ from } \mathcal{V}^{k,\infty} \text{ to } W^{k-1,\infty}(\mathbb{R}^N, \mathbb{R}^{2N}),$$

$$T \mapsto \det J_T \text{ from } \mathcal{V}^{k,\infty} \text{ to } W^{k-1,\infty}(\mathbb{R}^N, \mathbb{R})$$

at every point $T \in \mathcal{F}^{k,\infty}$ (J_T denotes here the standard Jacobian matrix of T).

For the proofs of (a) - (c) of the above lemma, we refer, respectively to Lemma 4.1 (see also [12]), Lemma 4.4 (i) and Lemma 4.3 and 4.2 of [15].

In what follows, we report on relationships between the convergence in $\mathcal{O}^{k,\infty}$ and other types of convergence of sets.

Let D be an open set of $\mathbb{R}^N$. By 1_D we will denote its characteristic function.

DEFINITION 1. By the Hausdorff complementary metric, we mean

$$d(\Omega_1, \Omega_2) = \max \left(\sup_{x \in D \setminus \Omega_1} \inf_{y \in D \setminus \Omega_2} \|x - y\|, \sup_{x \in D \setminus \Omega_2} \inf_{y \in D \setminus \Omega_1} \|x - y\| \right),$$

and the topology given by this metric we will denote by H^c.

REMARK 2. Let $k \geq 1$. Then
(i) If $\Omega_m \to \Omega_0$ in $\mathcal{O}^{k,\infty}$, then $1_{\Omega_m} \to 1_{\Omega_0}$ in $L^2(\mathbb{R}^N)$;
(ii) If $\Omega_m \to \Omega_0$ in $\mathcal{O}^{k,\infty}$ and $int\,\overline{C} = C$, then $\Omega_m \xrightarrow{H^c} \Omega_0$.

The following important property of the H^c convergence is the "covering" of the compacts.

REMARK 3. If $\Omega_m \xrightarrow{H^c} \Omega_0$, then

$$\forall\, G \subset\subset \Omega_0,\ \exists\, m_G \in \mathbb{N} \,:\, \forall\, m \geq m_G \;\; G \subseteq \Omega_m.$$

The following basic hypothesis will be needed in the next sections:

$\underline{H(C, \mathcal{B})}$: C is a bounded open set in $\mathbb{R}^N$ with boundary of class $W^{i,\infty}$, $i \geq 1$ such that $int\,\overline{C} = C$ and $\mathcal{B}$ is a bounded closed subset of $\mathcal{O}^{k,\infty}$, with $k \geq 3$ and $1 \leq i \leq k$.

3. Definitions and properties of some functional spaces

In this section we introduce some spaces of functions which will be useful in the sequel. Let I be a positive number, H a real Hilbert space, V a real Banach space, and V' its dual space. Suppose that $V \subset H$, V is dense in H and $\|u\|_H \leq \tilde{c}\|u\|_V$ for any $u \in V$, with $\tilde{c} > 0$. Identifying H with its dual space, we have $V \subset H \subset V'$. By $(\cdot,\cdot)_{V' \times V}$ we denote the pairing between V' and V. Let $L^2(0,I;V)$ be the space of functions v from $(0,I)$ to V, strongly measurable and square integrable with the norm:

$$\|v\|_{L^2(0,I;V)} = \left(\int_0^I \|v(t)\|_V^2 \; dt \right)^{\frac{1}{2}}.$$

The inclusion $V \subset H \subset V'$ implies $L^2(0,I;V) \subset L^2(0,I;H) \subset L^2(0,I;V')$, as $[L^2(0,I;V)]' \simeq L^2(0,I;V')$ (see [11]). Let $C(0,I;H)$ be the space of continuous functions from $[0,I]$ to H with the norm

$$\|v\|_{C(0,I;H)} = \sup_{0 \leq t \leq I} \|v(t)\|_H .$$

For $v \in L^2(0,I;V)$, denoting by v' vector-valued generalized derivative, we define the space

$$W(0,I;V) = \left\{ v : \; v \in L^2(0,I;V), \; v' \in L^2(0,I;V') \right\},$$

with the norm

$$\|v\|_{W(0,I;V)} = \left(\int_0^I \|v(t)\|_V^2 \; dt + \int_0^I \|v'(t)\|_{V'}^2 \; dt \right)^{\frac{1}{2}}.$$

The space $W(0,I;V)$ supplied with the natural scalar product (generating the above norm) is the real Hilbert space, which is continuously embedded in $C(0,I;H)$.

Some properties of the above functional spaces and their connections with the mapping method, are given in the following lemmas (for the proofs see [12], Section 2).

LEMMA 2. *Let V be a subspace of $H^1(\Omega)$. Suppose $T(C) = \Omega$, $T \in \mathcal{F}^{2,\infty}$, and put*

$$\underline{V} = \{ v \circ T : \; v \in V \} \subset H^1(C).$$

Then the operator $\mathbb{T}$: $f \longmapsto \widehat{f}_T$, where $\widehat{f}_T(t,X) = f(t,T(X))$, is an isomorphism from $L^2(0,I;V)$ to $L^2(0,I;\underline{V})$ and from $W(0,I;V)$ to $W(0,I;\underline{V})$. Furthermore, we have

$$\int_0^I \left(f'(t), \phi(t) \right)_{V' \times V} \; dt = \int_0^I \left(\widehat{f}'_T, \widehat{\phi}_T |\det J_T| \right)_{\underline{V}' \times \underline{V}} \; dt, \tag{3}$$

for every $f \in W(0,I;V)$ and $\phi \in L^2(0,I;V)$, where $\widehat{\phi}_T(t,X) = \phi(t,T(X))$.

LEMMA 3. *Let $\{\Omega_n\}$ be a sequence of sets from $\mathcal{O}^{k,\infty}$, let $T_n \in \mathcal{F}^{k,\infty}$ be such that $T_n(C) = \Omega_n$ and $u_{\Omega_n} \in W(0,I;H^1(\Omega_n))$. If $\{\|u_{\Omega_n}\|_{W(0,I;H^1(\Omega_n))}\}$ is bounded and $\{J_{T_n}\}$, $\{J_{T_n}^{-1}\}$ are bounded in $W^{1,\infty}(\mathbb{R}^N;\mathbb{R}^N)$, then $\{\|\widehat{u}_n\|_{W(0,I;H^1(C))}\}$ is bounded, where $\widehat{u}_n(t,X) = u_{\Omega_n}(t,T(X))$.*

LEMMA 4. *If $f, f_n \in L^2(\mathbb{R}^{N+1})$ and $f_n(t,x) \to f(t,x)$ strongly in $L^2(\mathbb{R}^{N+1})$, and $T_n - T \to 0$, $T_n^{-1} - T^{-1} \to 0$ in $W^{1,\infty}(\mathbb{R}^N;\mathbb{R}^N)$, then $f_n(t,T_n(X)) \to f(t,T(X))$ strongly in $L^2(\mathbb{R}^{N+1})$.*

4. Hemivariational inequality with nonlinear law in Ω

In this section we investigate a class of parabolic hemivariational inequalities with nonlinear laws appearing in Ω.

Let Ω be an open, bounded subset of $\mathbb{R}^N$. Let us introduce the following spaces: $V = V(\Omega) = H^1(\Omega)$, $H = H(\Omega) = L^2(\Omega)$, $\mathcal{V} = \mathcal{V}(\Omega) = L^2(0,I;V)$, $\mathcal{V}' = \mathcal{V}'(\Omega) = L^2(0,I;V')$, $\mathcal{H} = \mathcal{H}(\Omega) = L^2(0,I;H)$, $\mathcal{W} = \mathcal{W}(\Omega) = W(0,I;V) = \{v : v \in \mathcal{V},\ v' \in \mathcal{V}'\}$.

We suppose that $a : V \times V \to \mathbb{R}$ is defined by

$$a(u,v) = \int_\Omega [(A\nabla u, \nabla v) + a_0 uv]\ dx,$$

and satisfies the following hypothesis

$\underline{H(a)}$: The norm $a : V \times V \to \mathbb{R}$ is a bilinear, continuous (i.e. $|a(u,v)| \le M\|u\|\,\|v\|$ for $u,v \in V$ with $M > 0$), symmetric and coercive on V (i.e. $a(v,v) \ge \alpha\|v\|^2$ for $v \in V$ with $\alpha > 0$ independent of Ω), the matrix $A \in [C(\mathbb{R}^N)]^{N^2} \cap [L^\infty(\mathbb{R}^N)]^{N^2}$ and $a_0 \in C(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$, $a_0(x) \ge \tilde{a} > 0$ a.e. in $\mathbb{R}^N$.

Adopting the notation of [7] for a given $\beta \in L^\infty_{loc}(\mathbb{R})$ we denote by $\widehat{\beta}: \mathbb{R} \to 2^{\mathbb{R}}$ a multifunction obtained from β by filling in the gaps at its discontinuity points, i.e.

$$\widehat{\beta}(\xi) = [\underline{\beta}(\xi), \overline{\beta}(\xi)],$$

where

$$\underline{\beta}(\xi) = \lim_{\delta\to 0^+} \operatorname*{ess\,inf}_{|t-\xi|\le\delta} \beta(t), \quad \overline{\beta}(\xi) = \lim_{\delta\to 0^+} \operatorname*{ess\,sup}_{|t-\xi|\le\delta} \beta(t)$$

and $[\cdot,\cdot]$ denotes the interval. It is well known (cf. [2]) that a locally Lipschitz function $j: \mathbb{R} \to \mathbb{R}$ can be determined up to an additive constant by the relation $j(\xi) = \int_0^\xi \beta(s)\, ds$ and that $\partial j(\xi) \subset \widehat{\beta}(\xi)$. Moreover, if $\beta(\xi \pm 0)$ exist for every $\xi \in \mathbb{R}$, then $\partial j(\xi) = \widehat{\beta}(\xi)$. Here $\partial j: \mathbb{R} \to 2^{\mathbb{R}}$ denotes the Clarke's generalized subdifferential of j (see [3]) given by

$$\partial j(\xi) = \{\eta \in \mathbb{R} \mid j^0(\xi;\gamma) \ge \eta\gamma,\ \forall\, \gamma \in \mathbb{R}\} \quad \text{for all } \xi \in \mathbb{R}.$$

The notation $j^0(\cdot;\cdot)$ stands for the Clarke's directional derivative defined by

$$j^0(\xi;\gamma) = \limsup_{h\to 0,\ \tau\downarrow 0} \frac{j(\xi+h+\tau\gamma)-j(\xi+h)}{\tau} \quad \text{for all } \xi,\gamma\in\mathbb{R}.$$

We will also assume the hypotheses

$\underline{H(\beta)}$: The function $\beta \in L^\infty_{loc}(\mathbb{R})$ is such that

(i) $\beta(\xi \pm 0)$ exists for each $\xi \in \mathbb{R}$;

(ii) there exists $c_0 > 0$ such that $|\beta(\xi)| \le c_0(1+|\xi|)$ for $\xi \in \mathbb{R}$.

$\underline{H(f,\psi)}$: $f \in \mathcal{H}(\mathbb{R}^N)$, $\psi \in H(\mathbb{R}^N)$.

By an evolution hemivariational inequality we mean the following problem:

$$\text{(PHVI)} \quad \begin{cases} \text{find } u \in \mathcal{W} \text{ such that} \\ (u'(t), v-u(t))_{V'\times V} + a(u(t), v-u(t)) \\ \quad + \int_\Omega j^0(u(t), v-u(t))\, dx \ge (f(t), v-u(t))_{V'\times V}, \\ \quad \forall\, v\in V, \ \text{a.e. } t\in(0,I), \\ u(0) = \psi. \end{cases}$$

The concept of solution to problem (PHVI) is specified below.

DEFINITION 2. An element $u \in \mathcal{W}$ is said to be a solution to (PHVI) if there exists $\chi \in \mathcal{H}(\Omega)$ such that

$$\begin{cases} (u'(t), v) + a(u(t), v) + (\chi(t), v) \\ \quad = (f(t), v)\,, \quad \forall\, v\in V, \quad \text{a.e. } t\in(0,I), \\ u(0) = \psi \quad \text{in } \Omega, \\ \chi(t,x) \in \partial j(u(t,x)) \quad \text{a.e. } (t,x)\in(0,I)\times\Omega. \end{cases} \tag{4}$$

In the sequel, by $S(\Omega)$ we denote the set of all solutions to (PHVI).

The following existence result is due to Miettinen (see [13]).

THEOREM 2. *If hypotheses $H(a)$, $H(\beta)$ hold and $f \in \mathcal{V}'$, $\psi \in H(\mathbb{R}^N)$, then problem (PHVI) admits a solution, i.e. $S(\Omega) \ne \emptyset$.*

Due to the lack of convexity of j (or some additional growth condition on the function β, see [13]), no uniqueness result for (PHVI) can be obtained, so $S(\Omega)$ contains, in general, more than one element.

To simplify the notation for $w \in \mathcal{V}'$, $f \in \mathcal{H}(\mathbb{R}^N)$ and $u, v \in \mathcal{V}$ we put

$$\int_0^I (w,v)\ dt = \int_0^I (w(t),v(t))_{V'\times V}\ dt,$$
$$\int_0^I a(u,v)\ dt = \int_0^I \int_\Omega \big[(A(x)u(t,x),v(t,x)) + a_0(x)u(t,x)v(t,x)\big]\ dx\ dt,$$
$$\int_0^I (f,v)\ dt = \int_0^I \int_\Omega f(t,x)v(t,x)\ dx\ dt.$$

Moreover for $v \in \mathcal{V}$, $T \in \mathcal{F}^{k,\infty}$ instead of writing $\widehat{v}(t,x) = v(t,T(x))$ for all $t \in (0,I)$, we write $\widehat{v} = v \circ T$.
The following result will be crucial in the proof of the main theorem.

PROPOSITION 1. Let us assume that $H(C,\mathcal{B})$, $H(a)$, $H(\beta)$ and $H(f,\psi)$ hold. Then the map $\mathcal{B} \ni \Omega \mapsto S(\Omega) \subset \mathcal{W}$ has a closed graph in the following sense: if $\Omega_m, \Omega_0 \in \mathcal{B}$, $\Omega_m \to \Omega_0$ in $\mathcal{O}^{k,\infty}$, $u_m \in S(\Omega_m)$, $\widehat{u}_m = u_m \circ T_m$, $\widehat{u}_m \to u^*$ weakly in $\mathcal{W}(C)$, then $u^* = u_0 \circ T_0$ for some $u_0 \in S(\Omega_0)$, where $\Omega_m = T_m(C)$ and $\Omega_0 = T_0(C)$.

Proof. We follow some ideas of Liu and Rubio [12], as well as, of Denkowski and Migórski [7]. Let $\Omega_m, \Omega_0 \in \mathcal{B}$ be such that $\Omega_m \to \Omega_0$ in $\mathcal{O}^{k,\infty}$, where $\Omega_m = T_m(C)$ and $\Omega_0 = T_0(C)$. By definition $T_m, T_0 \in \mathcal{F}^{k,\infty}$ and $T_m - T_0 \to 0$, $T_m^{-1} - T_0^{-1} \to 0$ in $W^{k,\infty}(\mathbb{R}^N, \mathbb{R}^N)$. Without loss of generality, we suppose that $\det J_{T_m} > 0$ and $\det J_{T_0} > 0$ on $\mathbb{R}^N$. Let $u_m \in S(\Omega_m)$, i.e. $u_m \in \mathcal{W}$ and there exists $\chi_m \in \mathcal{H}(\Omega_m)$ such that

$$\begin{cases} \displaystyle\int_0^I (u_m'(t),v)\phi(t)\ dt + \int_0^I a(u_m(t),v)\phi(t)\ dt + \int_0^I (\chi_m(t),v)\phi(t)\ dt \\ \displaystyle\quad = \int_0^I (f(t),v)\phi(t)\ dt\ , \quad \forall\, v \in V, \quad \forall\, \phi \in \mathcal{D}\left((0,I)\right), \\ u_m(0) = \psi \quad \text{in } \Omega_m, \\ \chi_m(t,x) \in \partial j(u_m(t,x)) \quad \text{a.e. } (t,x) \in (0,I)\times\Omega_m. \end{cases} \tag{5}$$

By using the transformation $x = T_m(X)$, and applying Lemma 2 we rewrite (5) as the following equivalent problem on the set C:

$$\int_0^I (\widehat{u}_m', \widehat{v})\ dt + \int_0^I a_{T_m}(\widehat{u}_m, \widehat{v})\ dt + \int_0^I (\widehat{\chi}_m, \widehat{v})\ dt$$
$$= \int_0^I (\widehat{f}_m, \widehat{v})\ dt\ , \quad \forall\, \widehat{v} \in V(C), \quad \forall\, \phi \in \mathcal{D}\left((0,I)\right), \tag{6}$$
$$\widehat{u}_m(0,X) = \psi(T_m(X)) \quad \text{in } C, \tag{7}$$
$$\widehat{\chi}_m(t,X) \in \partial j(\widehat{u}_m(t,X)) \quad \text{a.e. } (t,X) \in (0,I)\times C, \tag{8}$$

where $\widehat{u}_m = u_m \circ T_m, \widehat{A}_{T_m} = A \circ T_m, \widehat{\chi}_m = \chi_m \circ T_m, \widehat{f}_m = f \circ T_m, \widehat{a}_m = a_0 \circ T_m,$ and

$$
\begin{aligned}
(\widehat{u}_m', \widehat{v}) &= \int_C \widehat{u}_m'(t,X)\widehat{v}(t,X)\phi(t) \det J_{T_m}(X)\ dX,\\
a_{T_m}(\widehat{u}_m, \widehat{v}) &= \int_C \Big[\Big(J_{T_m}^{-1}(X)\widehat{A}_{T_m}(X)J_{T_m}^{-t}(X)\nabla\widehat{u}_m(t,X), \nabla\widehat{v}(t,X)\Big)\\
&\qquad + \widehat{a}_m(X)\widehat{u}_m(t,X)\widehat{v}(t,X)\Big]\ \phi(t) \det J_{T_m}(X)\, dX,\\
(\widehat{\chi}_m, \widehat{v}) &= \int_C \widehat{\chi}_m(t,X)\widehat{v}(t,X)\ \phi(t) \det J_{T_m}(X)\ dX,\\
(\widehat{f}_m, \widehat{v}) &= \int_C \widehat{f}_m(t,X)\widehat{v}(t,X)\ \phi(t) \det J_{T_m}(X)\, dX.
\end{aligned}
$$

We may consider $\widehat{v}$ and ϕ in (6) to be fixed. Moreover, we know (see Lemma 2) that $\widehat{u}_m \in \mathcal{W}(C)$ and $\widehat{\chi}_m, \widehat{f}_m \in \mathcal{H}(C)$.

Our goal is to pass to the limit, as $m \to +\infty$, in the problem (6)–(8). By hypothesis

$$\widehat{u}_m \to u^* \quad \text{weakly in } \mathcal{W}(C), \tag{9}$$

i.e. $\widehat{u}_m \to u^*$ weakly in $\mathcal{V}(C)$ and

$$\widehat{u}_m' \to u^{*\prime} \quad \text{weakly in } \mathcal{V}'(C). \tag{10}$$

From (9) and the compactness of the embedding $\mathcal{W} \subset \mathcal{H}$, we have

$$\widehat{u}_m \to u^* \quad \text{in } \mathcal{H}(C). \tag{11}$$

On the other hand, by using $H(\beta)(ii)$, from (8) we get

$$
\begin{aligned}
\|\widehat{\chi}_m\|^2_{\mathcal{H}(C)} &= \int_0^I \int_C |\widehat{\chi}_m(t,X)|^2\ dX\ dt\\
&\le 2c_0^2 \int_0^I \int_C (1 + |\widehat{u}_m(t,X)|^2)\ dX\ dt\\
&\le c_1(\mathrm{m}(C) + \|\widehat{u}_m\|^2_{\mathcal{H}(C)}).
\end{aligned}
$$

Thus

$$\|\widehat{\chi}_m\|_{\mathcal{H}(C)} \le c_2(\sqrt{\mathrm{m}(C)} + \|\widehat{u}_m\|_{\mathcal{H}(C)}), \tag{12}$$

with $c_2 = c_2(c_0, I) > 0$, where $\mathrm{m}(C)$ denotes the Lebesgue measure of the set C. Hence and from (11), after passing to a subsequence if necessary, we have

$$\widehat{\chi}_m \to \chi^* \quad \text{weakly in } \mathcal{H}(C) \tag{13}$$

with $\chi^* \in \mathcal{H}(C)$.

By Lemma 4, we know that $\widehat{f}_m \to \widehat{f}_0$ in $\mathcal{H}(\mathbb{R}^N) = L^2(0,I;L^2(\mathbb{R}^N))$ with $\widehat{f}_0 = f \circ T_0$. It can be verified that

$$\int_0^I (\widehat{f}_m, \widehat{v})\ dt \to \int_0^I (\widehat{f}_0, \widehat{v})\ dt. \tag{14}$$

Indeed, we have

$$\begin{aligned}
&\left| \int_0^I (\widehat{f}_m, \widehat{v})\ dt - \int_0^I (\widehat{f}_0, \widehat{v})\ dt \right| \\
&= \left| \int_0^I \int_C \widehat{f}_m \widehat{v} \phi \det J_{T_m}\ dX\ dt - \int_0^I \int_C \widehat{f}_0 \widehat{v} \phi \det J_{T_0}\ dX\ dt \right| \\
&\leq \|\phi\| \, \| \det J_{T_m} - \det J_{T_0} \| \int_0^I \int_C \left| \widehat{f}_m \widehat{v} \right|\ dX\ dt \\
&\quad + \left| \int_0^I \int_C \left(\widehat{f}_m - \widehat{f}_0 \right) \widehat{v} \phi \det J_{T_0}\ dX\ dt \right|.
\end{aligned}$$

The first term on the right hand side converges to zero since the sequence $\{\widehat{f}_m\}$ is bounded in $\mathcal{H}(C)$ and $\det J_{T_m} \to \det J_{T_0}$ in $L^\infty(\mathbb{R}^N)$ (as a consequence of Lemma 1(c)). The second term on the right hand side also tends to zero due to the strong convergence of $\widehat{f}_m$ to $\widehat{f}_0$ in $\mathcal{H}(C)$.

In an analogous way as we proved (14), we can show, using Lemma 1 and (13) that

$$\int_0^I (\widehat{\chi}_m, \widehat{v})\ dt \to \int_0^I (\chi^*, \widehat{v})\ dt. \tag{15}$$

Subsequently, from the assumptions on the matrix A, we deduce that $A(\cdot)$ is uniformly continuous on every bounded subset of $\mathbb{R}^N$. Since $T_m \to T_0$, $T_m^{-1} \to T_0^{-1}$ in $W^{k,\infty}(C;\mathbb{R}^N)$ and $T_m(C)$, $T_0(C)$ are in a bounded set of $\mathbb{R}^N$, we obtain

$$\widehat{A}_{T_m} \to \widehat{A}_{T_0} \quad \text{in } [L^\infty(C)]^{N^2}.$$

Hence and from Lemma 1, we have

$$J_{T_m}^{-1} \widehat{A}_{T_m} J_{T_m}^{-t} \to J_{T_0}^{-1} \widehat{A}_{T_0} J_{T_0}^{-t} \quad \text{in } [L^\infty(C)]^{N^2}. \tag{16}$$

As $\widehat{a}_m \to \widehat{a}_0$ in $L^\infty(\mathbb{R}^N)$, where $\widehat{a}_0 = a_0 \circ T_0$, so from the following inequality

$$\begin{aligned}
&\left| \int_0^I a_{T_m}(\widehat{u}_m, \widehat{v})\ dt - \int_0^I a_{T_0}(\widehat{u}_m, \widehat{v})\ dt \right| \\
&\leq \left| \int_0^I \int_C \left(\left[J_{T_m}^{-1} \widehat{A}_{T_m} J_{T_m}^{-t} - J_{T_0}^{-1} \widehat{A}_{T_0} J_{T_0}^{-t} \right] \nabla \widehat{u}_m, \nabla \widehat{v} \right) \phi \det J_{T_m}\ dX\ dt \right|
\end{aligned}$$

$$
\begin{aligned}
&+\left|\int_0^I\int_C \left(J_{T_0}^{-1}\widehat{A}_{T_0}J_{T_0}^{-t}\nabla\widehat{u}_m,\nabla\widehat{v}\right)\ \phi\left[\det J_{T_m}-\det J_{T_0}\right]\,dX\ dt\right| \\
&+\left|\int_0^I\int_C (\widehat{a}_m-\widehat{a}_0)\,\widehat{u}_m\,\widehat{v}\,\phi\,\det J_{T_m}\ dX\ dt\right| \\
&+\left|\int_0^I\int_C \widehat{a}_0\widehat{u}_m\widehat{v}\phi\left(\det J_{T_m}-\det J_{T_0}\right)\ dX\ dt\right| \\
\leq{}& \|\phi\|\,\|\det J_{T_m}\|_{L^\infty(\mathbb{R}^N)}\,\|J_{T_m}^{-1}\widehat{A}_{T_m}J_{T_m}^{-t} \\
&-J_{T_0}^{-1}\widehat{A}_{T_0}J_{T_0}^{-t}\|\,\|\widehat{u}_m\|_{\mathcal{W}(C)}\,\|\widehat{v}\|_{V(C)} \\
&+\|\phi\|\,\|J_{T_0}^{-1}\widehat{A}_{T_0}J_{T_0}^{-t}\|\,\|\det J_{T_m}-\det J_{T_0}\|\,\|\widehat{u}_m\|_{\mathcal{W}(C)}\,\|\widehat{v}\|_{V(C)} \\
&+\|\phi\|\,\|\det J_{T_m}\|_{L^\infty(\mathbb{R}^N)}\,\|\widehat{a}_m-\widehat{a}_0\|_{L^\infty(\mathbb{R}^N)}\,\|\widehat{u}_m\|_{\mathcal{W}(C)}\,\|\widehat{v}\|_{V(C)} \\
&+\|\phi\|\,\|\det J_{T_m}-\det J_{t_0}\|_{L^\infty(\mathbb{R}^N)}\,\|\widehat{a}_0\|_{L^\infty(\mathbb{R}^N)}\,\|\widehat{u}_m\|_{\mathcal{W}(C)}\,\|\widehat{v}\|_{V(C)},
\end{aligned}
$$

by taking (9), (16) and Lemma 1 into account, we get

$$
\int_0^I a_{T_m}(\widehat{u}_m,\widehat{v})\ dt-\int_0^I a_{T_0}(\widehat{u}_m,\widehat{v})\ dt\to 0. \tag{17}
$$

From the following inequality

$$
\begin{aligned}
&\left|\int_0^I a_{T_m}(\widehat{u}_m,\widehat{v})\ dt-\int_0^I a_{T_0}(u^*,\widehat{v})\ dt\right| \\
&\quad\leq\left|\int_0^I a_{T_m}(\widehat{u}_m,\widehat{v})\ dt-\int_0^I a_{T_0}(\widehat{u}_m,\widehat{v})\ dt\right|+ \\
&\quad\quad+\left|\int_0^I a_{T_0}(\widehat{u}_m,\widehat{v})\ dt-\int_0^I a_{T_0}(u^*,\widehat{v})\ dt\right|,
\end{aligned}
$$

by using (17) and weak-$\mathcal{V}$ continuity of the function $\mathcal{V}\ni w\mapsto\int_0^I a(w,\widehat{v})\ dt$ (as it is linear and strongly continuous), we obtain

$$
\int_0^I a_{T_m}(\widehat{u}_m,\widehat{v})\ dt\to\int_0^I a_{T_0}(u^*,\widehat{v})\ dt. \tag{18}
$$

Now, owing to (14), (15), (10), (18), we can pass to the limit in (6) and get

$$
\begin{aligned}
&\int_0^I (u^{*\prime},\widehat{v})\ dt+\int_0^I a_{T_0}(u^*,\widehat{v})\ dt+\int_0^I(\chi^*,\widehat{v})\ dt \\
&\quad=\int_0^I(\widehat{f}_0,\widehat{v})\ dt,\quad\forall\,\widehat{v}\in V(C),\quad\forall\phi\in\mathcal{D}\left((0,I)\right).
\end{aligned} \tag{19}
$$

In order to pass to the limit in (7) we observe that the operator $\mathcal{W}(C)\ni u\longmapsto u(0)\in H(C)$ is linear and continuous (which is a consequence of the continuous embedding of $\mathcal{W}(C)$ in $C(0,I;H(C))$). Therefore it is continuous with respect to weak topologies and from (9) we have: $\widehat{u}_m(0) \to u^*(0)$ weakly in $H(C)$. On the other hand, we have $\psi(T_m(X)) \to \psi(T_0(X))$ in $H(\mathbb{R}^N)$, so from the uniqueness of the weak limit, we get

$$u^*(0,X) = \psi(T_0(X)) \quad \text{in } C. \tag{20}$$

By passing to subsequences, if necessary, from (11) and (13), we have

$$\begin{aligned} &\widehat{u}_m \to u^* \ \text{ a.e. in } (0,I)\times C,\\ &\widehat{\chi}_m \to \chi^* \ \text{ weakly in } L^1((0,I)\times C). \end{aligned}$$

Since the multifunction $\partial j(\cdot)$ is u.s.c. with nonempty, convex and compact values (see [3]), by exploiting the above convergences, and applying the convergence theorem (see [1], p.273), we deduce from (8) that

$$\chi^*(t,X) \in \partial j(u^*(t,X)) \quad \text{a.e. } (t,X)\in(0,I)\times C. \tag{21}$$

Now we write down the problem (19)–(21) in an equivalent form by employing the transformation $X = T_0^{-1}(x)$. To this end, we introduce functions $u_0 = u^* \circ T_0^{-1}$ and $\chi_0 = \chi^* \circ T_0^{-1}$. From the relations $J_{T_0^{-1}}(x) = J_{T_0}^{-1}(T_0^{-1}(x))$ and $\det J_{T_0}(T_0^{-1}(x)) \cdot \det J_{T_0^{-1}}(x) = 1$ a.e. on $\mathbb{R}^N$ (cf. respectively, Corollary 2.1 and page IV-7 of [15]), we have

$$\begin{cases} \displaystyle\int_0^I (u_0'(t),v)\phi\,dt + \int_0^I a(u_0(t),v)\phi\,dt + \int_0^I (\chi_0(t),v)\phi\,dt \\ \quad \displaystyle = \int_0^I (f(t),v)\phi\,dt, \quad \forall\, v\in V(\Omega_0), \quad \forall\,\phi\in\mathcal{D}\left((0,I)\right), \\ u(0)=\psi \quad \text{in } \Omega_0, \\ \chi_0(t,x)\in\partial j(u_0(t,x)) \quad \text{a.e. } (t,x)\in(0,I)\times\Omega_0. \end{cases}$$

As it is true for every $\phi \in \mathcal{D}\left((0,I)\right)$, so we also have

$$\begin{aligned} &(u_0'(t),v) + a(u_0(t),v) + (\chi_0(t),v) = (f(t),v),\\ &\quad \forall\, v\in V(\Omega_0), \quad \text{a.e. } t\in(0,I), \end{aligned}$$

(cf. e.g. [10] Chapter III, or [8], Chapter IV,4, Lemma 1.7). Since $u^* \in \mathcal{W}(C)$ and $\chi_0 \in \mathcal{H}(\Omega_0)$, from Lemma 2 we conclude that $u_0 \in S(\Omega_0)$ and $u^* = u_0 \circ T_0$. This completes the proof of the proposition. □

We will also need the following lemma.

LEMMA 5. *Let us assume that $H(C,\mathcal{B})$, $H(a)$, $H(\beta)$ hold and $f \in \mathcal{V}'$, $\psi \in H$. If $u \in S(\Omega)$, then the following estimate holds:*

$$\|u\|_{\mathcal{W}} \le b\,(1+\mathrm{m}(\Omega))\, e^{d(1+\mathrm{m}(\Omega))} \Big(1+\mathrm{m}(\Omega) + \|\psi\|_{L^2(\Omega)} + \|f\|_{\mathcal{V}'(\Omega)}\Big) \tag{22}$$

with constants $b, d > 0$ depending only on α, M, I, c_0, and not depending on Ω.

Proof. Let $u \in S(\Omega)$. So there exists a function $\chi \in \mathcal{H}(\Omega)$ such that (4) holds. Using $H(\beta)(ii)$, we easily find that

$$\|\chi(t)\|_{L^2(\Omega)} \leq c_2 \left(1 + \mathrm{m}(\Omega) + \|u(t)\|_{L^2(\Omega)}\right), \quad \text{a.e. } t \in (0, I), \tag{23}$$

with $c_2 > 0$. It can be shown that:

$$\left|\int_0^t (\chi, u)\ ds\right| \leq c_3\left(1+\mathrm{m}(\Omega)\right) \cdot \left(1+\int_0^t \|u(s)\|^2_{L^2(\Omega)}\ ds\right), \quad \forall t \in (0, I), \tag{24}$$

with $c_3 > 0$. Indeed, using $H(\beta)(ii)$ and Hölder's inequality, for an arbitrary $t \in (0, I)$, we have:

$$\begin{aligned}
\left|\int_0^t (\chi, u)\ ds\right| &\leq c_2 \int_0^t \int_\Omega (1 + |u(s,x)|)|u(s,x)|\ dx\ ds \\
&= c_2 \int_0^t \int_\Omega |u(s,x)|\ dx\ ds + c_2 \int_0^t \int_\Omega |u(s,x)|^2\ dx\ ds \\
&\leq c_2 \sqrt{I \cdot \mathrm{m}(\Omega)} \cdot \sqrt{\int_0^t \int_\Omega |u(s,x)|^2\ dx\ ds} + c_2 \int_0^t \|u(s)\|^2_{L^2(\Omega)}\ ds \\
&\leq c_3\left(1 + \mathrm{m}(\Omega)\right) \cdot \left(1 + \int_0^t \|u(s)\|^2_{L^2(\Omega)}\ ds\right).
\end{aligned}$$

Now, using integration by parts, coerciveness of the form a, Young's inequality ($ab \leq \frac{\alpha}{2}a^2 + \frac{1}{2\alpha}b^2$ for $a, b, \alpha > 0$) and (24), for an arbitrary $t \in (0, I)$, we have:

$$\begin{aligned}
&\frac{1}{2}\|u(t)\|^2_{L^2(\Omega)} - \frac{1}{2}\|u(0)\|^2_{L^2(\Omega)} \\
&= \int_0^t \frac{d}{ds}\frac{1}{2}\|u(s)\|^2_{L^2(\Omega)}\ ds = \int_0^t (u'(s), u(s))_{V' \times V}\ ds \\
&= \int_0^t -a(u(s), u(s))\ ds + \int_0^t -(\chi(s), u(s))_{L^2(\Omega)}\ ds \\
&\quad + \int_0^t (f(s), u(s))_{V' \times V}\ ds \leq -\alpha\|u\|^2_{L^2(0,t,V)} \\
&\quad + c_3\left(1 + \mathrm{m}(\Omega)\right) \cdot \left(1 + \int_0^t \|u(s)\|^2_{L^2(\Omega)}\ ds\right) \\
&\quad + \|f\|_{L^2(0,t,V')}\|u\|_{L^2(0,t,V)} \\
&\leq -\frac{\alpha}{2}\|u\|^2_{L^2(0,t,V)} + c_3\left(1 + \mathrm{m}(\Omega)\right) \\
&\quad + c_3\left(1 + \mathrm{m}(\Omega)\right)\int_0^t \|u(s)\|^2_{L^2(\Omega)}\ ds + \frac{1}{2\alpha}\|f\|^2_{L^2(0,t,V')}.
\end{aligned}$$

Hence, for $t \in (0, I)$ we have:

$$\frac{1}{2}\|u(t)\|^2_{L^2(\Omega)} + \frac{\alpha}{2}\|u\|^2_{L^2(0,t,V)}$$

$$\leq c_4 \left[1 + \mathrm{m}(\Omega) + \|\psi\|^2_{L^2(\Omega)} + \|f\|^2_{\mathcal{V}'}\right] + c_3 \left(1 + \mathrm{m}(\Omega)\right) \int_0^t \|u(s)\|^2_{L^2(\Omega)}\, ds. \tag{25}$$

Using Gronwall's inequality, we get:

$$\|u(t)\|^2_{L^2(\Omega)} \leq c_5 \left(1 + \mathrm{m}(\Omega) + \|\psi\|^2_{L^2(\Omega)} + \|f\|^2_{\mathcal{V}'}\right) e^{c_6(1+\mathrm{m}(\Omega))}, \quad t \in (0, I) \tag{26}$$

so from (25), (26), we have:

$$\|u\|_{\mathcal{V}} \leq c_7 \left(1 + \mathrm{m}(\Omega)\right) e^{c_6(1+\mathrm{m}(\Omega))} \left(1 + \mathrm{m}(\Omega) + \|\psi\|_{L^2(\Omega)} + \|f\|_{\mathcal{V}'}\right). \tag{27}$$

Now, we estimate $\|u'\|_{\mathcal{V}'}$ as follows:

$$\begin{aligned}
\|u'(t)\|_{V'} &= \sup_{v\in V, \|v\|_V=1} (u'(t), v)_{V'\times V} \\
&= \sup_{v\in V, \|v\|_V=1} \left[-a(u(t), v) - (\chi(t), v)_{L^2(\Omega)} + (f(t), v)_{V'\times V}\right] \\
&\leq \sup_{v\in V, \|v\|_V=1} \left(-a(u(t), v)\right) + \sup_{v\in V, \|v\|_V=1} \left(-(\chi(t), v)_{L^2(\Omega)}\right) \\
&\quad + \sup_{v\in V, \|v\|_V=1} (f(t), v)_{V'\times V} \\
&\leq M \sup_{v\in V, \|v\|_V=1} \|u(t)\|_V \|v\|_V + \sup_{v\in V, \|v\|_V=1} \|\chi(t)\|_{L^2(\Omega)} \|v\|_{L^2(\Omega)} \\
&\quad + \sup_{v\in V, \|v\|_V=1} \|f(t)\|_{V'} \|v\|_V \\
&\leq M\|u(t)\|_V + c_2 \left(1 + \mathrm{m}(\Omega) + \|u(t)\|_{L^2(\Omega)}\right) + \|f(t)\|_{V'} \\
&\leq c_8 \|u(t)\|_V + c_2 (1 + \mathrm{m}(\Omega)) + \|f(t)\|_{V'},
\end{aligned}$$

where we used the continuity of the form a, the inequality (23) and the fact that $\|u(t)\|_{L^2(\Omega)} \leq \|u(t)\|_{V(\Omega)}$. Now, after integrating both sides of the above inequality, we obtain:

$$\|u'\|_{\mathcal{V}'} \leq c_9 \left(1 + \mathrm{m}(\Omega) + \|u\|_{\mathcal{V}} + \|f\|_{\mathcal{V}'}\right), \tag{28}$$

and using (27), we have:

$$\|u'\|_{\mathcal{V}'} \leq c_{10} \left(1 + \mathrm{m}(\Omega)\right) e^{c_6(1+\mathrm{m}(\Omega))} \left(1 + \mathrm{m}(\Omega) + \|\psi\|_{L^2(\Omega)} + \|f\|_{\mathcal{V}'}\right). \tag{29}$$

From (28), (29) and the definition of the norm $\|u\|_{\mathcal{W}}$, we get (22). □

5. A shape optimization problem

In this section we consider the control problem governed by parabolic hemivariational inequality.

Let hypothesis $H(C,\mathcal{B})$ hold. By J we will denote the cost functional of the form:

$$J(\Omega, u) = \int_0^I \int_\Omega L(t,x,u)\, dx\, dt, \tag{30}$$

and by optimal shape design problem we mean the following problem:

$$(OSDP) \qquad \begin{cases} \text{Find } \Omega^* \in \mathcal{B} \text{ and } u^* \in S(\Omega^*) \text{ such that} \\ J(\Omega^*, u^*) = \min\limits_{\Omega\in\mathcal{B}} \min\limits_{u\in S(\Omega)} J(\Omega, u). \end{cases}$$

To obtain our main existence result for the solution of (OSDP), we need the additional hypothesis

$\underline{H(J)}$: $\quad J$ is l.s.c. with respect to the local convergence in $\mathbb{R}^{N+1}$,

where the local convergence is defined as follows (see [19]).

DEFINITION 3. Let D be an open subset of $\mathbb{R}^N$ and let ϕ_m, ϕ be locally summable functions defined in $\mathbb{R}^N$. We say that ϕ_m converges locally to ϕ in D if for any compact subset G of D, we have: ϕ_m is defined and summable on G, at least for m sufficiently large and $\|\phi_m - \phi\|_{L^1(G)} \to 0$, as $m \to +\infty$.

THEOREM 3. *If hypotheses $H(C,\mathcal{B})$, $H(a)$, $H(\beta)$, $H(J)$ and $H(f,\psi)$ hold, then (OSDP) admits at least one solution.*

Proof. We apply the direct method of the calculus of variations. Let (Ω_m, u_m) be a minimizing sequence for $(OSDP)$. From Theorem 1, as $\mathcal{B}$ is compact in $\mathcal{O}^{k-1,\infty}$, we can choose a subsequence of Ω_m (still indexed by m) and a set $\Omega_0 \in \mathcal{B}$ such that $\Omega_m \to \Omega_0$ in $\mathcal{O}^{k-1,\infty}$. This means that there exist T_m, $T_0 \in \mathcal{F}^{k-1,\infty}$ such that $\Omega_m = T_m(C)$, $\Omega_0 = T_0(C)$ and $T_m - T_0 \to 0$, $T_m^{-1} - T_0^{-1} \to 0$ in $W^{k-1,\infty}(\mathbb{R}^N, \mathbb{R}^N)$.

Since $u_m \in S(\Omega_m)$, from Lemma 5, we have

$$\|u_m\|_{\mathcal{W}(\Omega_m)} \\ \leq b\,(1+\mathrm{m}(\Omega_m))\, e^{d(1+\mathrm{m}(\Omega_m))} \Big(1+\mathrm{m}(\Omega_m)+\|\psi\|_{L^2(\Omega_m)}+\|f\|_{\mathcal{H}(\Omega_m)}\Big). \tag{31}$$

From Remark 2, we have $1_{\Omega_m} \to 1_{\Omega_0}$ in $L^2(\mathbb{R}^N)$ which gives, in particular, that $\{\mathrm{m}(\Omega_m)\}$ are bounded, so also $\{\|\psi\|_{L^2(\Omega_m)}\}$ and $\{\|f\|_{\mathcal{H}(\Omega_m)}\}$ are bounded. Therefore from (31) we can see that $\{\|u_m\|_{\mathcal{W}(\Omega_m)}\}$ lies in a bounded set in $\mathbb{R}$.

Putting $\widehat{u}_m = u_m \circ T_m$ and using Lemma 3, we obtain that $\{\|\widehat{u}_m\|_{\mathcal{W}(C)}\}$ are bounded. Thus, taking next subsequence if necessary, we have

$$\widehat{u}_m \to u^* \quad \text{weakly in } \mathcal{W}(C) \tag{32}$$

with some $u^* \in \mathcal{W}(C)$. From Proposition 1we have $u^* = u_0 \circ T_0$ and $u_0 \in S(\Omega_0)$. So the pair (Ω_0, u_0) is admissible for $(OSDP)$.

Let $\underline{\widehat{u}_m}$ and $\underline{u^*}$ denote the functions in $\mathcal{W}(\mathbb{R}^N)$ obtained from $\widehat{u}_m$ and u^*, respectively, by extending them by zero outside C. From (32) and the compactness of the embedding $\mathcal{W}(C) \subset \mathcal{H}(C)$, we get

$$\underline{\widehat{u}_m} \to \underline{u^*} \quad \text{in } \mathcal{H}(\mathbb{R}^N).$$

So from Lemma 4, we also have

$$\underline{u_m} \to \underline{u_0} \quad \text{in } \mathcal{H}(\mathbb{R}^N), \tag{33}$$

where

$$\underline{u_m}(t,x) = \begin{cases} u_m(t,x), & \text{if } x \in \Omega_m \\ 0, & \text{if } x \in \mathbb{R}^N \setminus \Omega_m, \end{cases}$$

$$\underline{u_0}(t,x) = \begin{cases} u_0(t,x), & \text{if } x \in \Omega_0 \\ 0, & \text{if } x \in \mathbb{R}^N \setminus \Omega_0. \end{cases}$$

On the other hand, from Remarks 2 and 3, we deduce that for any compact G in Ω_0, there is an $m_G > 0$ such that $G \subset \Omega_m$ for all $m \geq m_G$. Now, from (33), we can see that for any such G we have $\|u_m - u_0\|_{\mathcal{H}(G)} \to 0$, and, in consequence, $u_m \to u_0$ locally in $\mathbb{R}^{N+1}$. Hence, due to the hypothesis $H(J)$, we conclude that (Ω_0, u_0) solves the problem $(OSDP)$. □

6. Comments and applications

As an application of $(OSDP)$ governed by $(PHVI)$ we would like to mention the semipermeability problems which were first studied by Duvaut and Lions (see [4]) for a monotone semipermeability relations and led to variational inequalities. The generalizations of these problems (without assuming monotonicity) was studied by Panagiotopoulos (see [18]) and led to hemivariational inequalities. Two main classes of semipermeability problems may be considered: the interior and the boundary semipermeability problems (see [4]). In the first class, for instance, we seek a function u such as to satisfy

$$\frac{\partial u}{\partial t} - \Delta u = f \quad \text{in } \Omega \times [0, I],$$

with

$$f = \overline{f} + \overline{\overline{f}}, \quad -\overline{f} \in \partial j(u) \quad \text{in } \Omega \times [0, I],$$

where j is a superpotential (in the sense of [18]) which can be of nonmonotone, nonconvex type, possibly multivalued. Function u is also supposed to satisfy the classical boundary condition

$$u = 0 \quad \text{on } \partial\Omega \times [0, I],$$

as well as the initial condition

$$u|_{t=0} = u_0.$$

For possible choices of j, we refere to [18] (pp 30, fig.1), where a control problem of temperature regulation by thermostatic devices is considered.

REMARK 4. The typical cost functional of the form (30) arising in heat conduction problems, hydraulics and electrostatics is the following:

$$J(\Omega, u) = \int_0^I \int_\Omega |u(x,t) - u_0(x,t)|^2 \, dx \, dt.$$

Here t denotes the time and u represents the temperature in the case of heat conduction problems, the pressure in hydraulics problems and the electric potential in electrostatics.

The lower semicontinuity of the above functional with respect to the local convergence was obtained by Denkowski and Migórski [6] (without employing the methods used by Serrin [19]).

REMARK 5. Liu and Rubio [12, part 2] studied $(OSDP)$ for variational inequality of parabolic type of the form

$$\begin{cases} \min\limits_{\Omega \in \mathcal{B}} \int_0^I \int_\Omega L(t,x,u) \, dx \, dt \quad \text{such that} \\ (u'(t), v - u(t))_{V' \times V} + a(u(t), v - u(t)) \\ \quad \geq (f(t), v - u(t))_{V' \times V}, \quad \forall \, v \in K, \quad \text{a.e. } t \in (0, I), \\ u(0) = \psi, \end{cases}$$

where K is a closed, convex, nonempty subset of V. Theorem 3 extends their result on existence of optimal shapes to the case of $(PHVI)$ with $K = V$. In this extension the main difficulty consists in the fact that, in general, (PHVI) posseses many solutions. This leads us to the investigation of the closedness of the map which to every admissible shape assigns the solution set of $(PHVI)$ (see Proposition 1)

REMARK 6. It is possible to consider the cost functional of the more general form

$$J(\Omega, u) = \int_0^I \int_\Omega L(t, x, u, \nabla u)\ dx\ dt.$$

The sufficient conditions for lower semicontinuity of the functional J with respect to the local convergence were given by Serrin in [19], for instance, the integrand $L(t, x, u, p)$ should be nonnegative, continuous in (t, x, u, p) and strictly convex in p.

REMARK 7. Another natural extension of our result leads us to the $(PHVI)$ of the form

$$(u'(t) + Au(t), v - u(t))_{V' \times V} + \int_\Omega j^0(u(t), v - u(t))\ dx$$
$$\geq (f(t), v - u(t))_{V' \times V}, \quad \forall\, v \in V, \quad \text{a.e. } t \in (0, I),$$

with a nonlinear operator $A : V \longmapsto V'$. In this case the existence problem for (OSDP) seems to be open.

REMARK 8. In order to incorporate various unilateral conditions on Ω or on $\partial\Omega$, one have to study $(OSDP)$ for $(PHVI)$ considered in a close, convex, nonempty subset K of V. In this case the existence of solutions of $(PHVI)$ is also an open problem (in [13] only the case $K = V$ was studied).

REMARK 9. Similarly, we can deal with $(OSDP)$ for $(PHVI)$ where the non-monotone, multivalued law is prescribed on the boundary of Ω. The treatment of such $(PHVI)$ is analogous to paper [7].

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